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# Quantum Measurement II

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Some of the basic results of the quantum theory of measurement are reviewed and an application of the theory of sequential measurements to a determination of a geometric phase in a measurement cycle is discussed.

## 1. INTRODUCTION

The quantum theory of measurement is the theory of a measurement process within quantum mechanics. This theory was initiated by von Neumann (1932), and his formulation of the measurement process within the theory of compound systems in quantum mechanics has become a paradigm for measurements (Jauch, 1968; Beltrametti and Cassinelli, 1981; Wheeler and Zurek, 1983). However, in recent years von Neumann's model has been extended in various ways to cover whole classes of measurements of arbitrary observables (as POV measures). The systematic theory of measurement which has thus emerged has opened up a possibility of investigating the consistency of the interpretation of quantum mechanics as a theory of individual objects (see, e.g., Mittelstaedt, 1993), of studying various properties of measurements, such as repeatability, ideality, and correlations, and of working out experimentally testable consequences of the measurement theory. All this development has been greatly stimulated by the advanced opportunities for experimenting with individual atomic objects, as demonstrated by many ingenious experiments, such as the correlation experiments on pairs of photons, or the interferometer experiments with individual neutrons, or the various trapping experiments with atomic particles. For the topic of the present paper it is particularly impressive to note that, for instance, the continuous Stern–Gerlach effect

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allows one "to perform and *repeat* a quantum measurement on the same individual atomic particle, as often as one pleases" (Dehmelt, 1990).

The aim of my paper is to present an overview of some of the main achievements in the systematic development of the quantum theory of measurement, and to present an application of this theory to a determination of the geometric phase accumulating in a sequence of measurements on an individual object.

To establish a common language and notations, I shall start with a brief summary of the most relevant aspects of the Hilbert space description of quantum mechanics, based on the dual concepts of states and observables, and their probability measures.

Let $\mathscr{H}$ be a (complex, separable) Hilbert space representing a physical system $\mathscr{S}$. Let $\mathscr{L}(\mathscr{H})$ stand for the set of bounded linear operators on $\mathscr{H}$, and let $\mathscr{T}(\mathscr{H})$ denote its subset of trace class operators. An *observable* of $\mathscr{S}$ is represented as (and identified with) a positive operator valued measure, a POV measure $E: \mathscr{F} \to \mathscr{L}(\mathscr{H})$, defined on a measurable space $(\Omega, \mathscr{F})$, the space of values of $E$. Usually, $(\Omega, \mathscr{F})$ is the real Borel space $(\mathfrak{R}^n, \mathscr{B}(\mathfrak{R}^n))$ for some $n = 1, 2, \ldots$. Among the observables are those represented by projection operator valued measures, PV measures. They can be identified with self-adjoint operators $A$ (acting in $\mathscr{H}$), provided that the value space of the observable is just the usual one, $(\mathfrak{R}, \mathscr{B}(\mathfrak{R}))$. A *state* of $\mathscr{S}$ is represented as (and identified with) a positive trace-one operator $T \in \mathscr{T}(\mathscr{H})_1^+$. Again, among the states there are the pure states $P[\varphi]$, or the vector states $\varphi$, defined by the unit vectors $\varphi$ of $\mathscr{H}$ (as $P[\varphi]\psi := \langle \varphi | \psi \rangle \varphi$, $\psi \in \mathscr{H}$).

The important structure is that any observable $E$ and state $T$ define a probability measure $p_T^E$ on the space of the values of $E$,

$$p_T^E(X) := \mathrm{tr}[TE(X)], \qquad X \in \mathscr{F} \tag{1}$$

for which the *minimal interpretation* is adopted: the number $p_T^E(X)$ is the probability that a measurement of $E$ on the system in the state $T$ leads to a result in the set $X$.

Since I shall refer frequently to an observable given by a discrete self-adjoint operator, let me fix here also this special case. Let $A = \sum a_i P_i$ be a discrete self-adjoint operator with the eigenvalues $a_i$ and the associated spectral projections $P_i$. The important feature of such an operator is that there are orthonormal bases $\{\varphi_{ij}\}$ of $\mathscr{H}$ consisting of eigenvectors of $A$; $A\varphi_{ij} = a_i \varphi_{ij}$, with the second index counting for the (possible) degeneracy of the eigenvalue $a_i$. If $\varphi$ is a (vector) state of $\mathscr{S}$, then $\varphi = \sum_{ij} \langle \varphi_{ij} | \varphi \rangle \varphi_{ij}$, and one gets

$$p_\varphi^A(a_i) = \sum_j |\langle \varphi_{ij} | \varphi \rangle|^2 \tag{2}$$

as the probability that a measurement of $A$ would lead to the result $a_i$.

## 2. MEASUREMENT THEORY

The quantum theory of measurement, as presented here, is an obvious (though nontrivial) generalization of von Neumann's (1932) model. Hence I shall recall first the construction of this paradigm. As expected, von Neumann's model turns out to be an important special case of a repeatable measurement of an observable represented as a discrete self-adjoint operator.

### 2.1. The Measurement Scheme

Consider an observable given by a discrete self-adjoint operator $A = \sum a_i P_i$. To construct a measurement model for this observable, one usually starts by choosing an orthonormal basis $\{\varphi_{ij}\}$ for $\mathscr{H}$ consisting of eigenvectors of $A$. Consider then an apparatus $\mathscr{A}$ represented by a Hilbert space $\mathscr{H}_{\mathscr{A}}$ of the dimension equal to the number of distinct eigenvalues $a_i$ of $A$, and let $\{\phi_i\}$ be any of its basis. Choose a pointer observable $Z = \sum_i z_i P[\phi_i]$. Let $\phi$ be the initial state of the apparatus, and define a unitary mapping $U_{vNL}$ via the relation

$$U_{vNL}(\varphi \otimes \phi) = \sum \langle \varphi_{ij} | \varphi \rangle \varphi_{ij} \otimes \phi_i \tag{3}$$

The mapping $U_{vNL}$, which acts on the tensor product Hilbert space $\mathscr{H} \otimes \mathscr{H}_{\mathscr{A}}$ of the compound object–apparatus system $\mathscr{S} + \mathscr{A}$, is meant to represent a measurement interaction between $\mathscr{S}$ and $\mathscr{A}$ correlating the values of the measured observable $A$ with those of the pointer observable $Z$. If $\varphi$ is the state of $\mathscr{S}$ before the measurement, then $U_{vNL}(\varphi \otimes \phi)$ is the state of $\mathscr{S} + \mathscr{A}$ after the measurement. In particular, $U_{vNL}(\varphi \otimes \phi)$ determines the state of the apparatus after the measurement as the reduced state $W \hat{=} \sum p_\varphi^A(a_i) P[\phi_i]$. It will now be immediate to observe that the measurement model defined by the items $\mathscr{H}_{\mathscr{A}}$, $Z$, $\phi$, and $U_{vNL}$ satisfies the condition

$$\text{if} \quad p_\varphi^A(a_i) = 1 \quad \text{then also} \quad p_W^Z(z_i) = 1 \tag{4}$$

This condition is equivalent to the seemingly stronger condition

$$p_\varphi^A(a_i) = p_W^Z(z_i) \tag{5}$$

for all $i = 1, 2, \ldots$ and for all initial states $\varphi$ of $\mathscr{S}$. This is the measurement model of von Neumann; it is a prototype of measurements often called von Neumann–Lüders or Lüders measurements.

Consider now an arbitrary observable $E$, represented as a POV measure. If $T$ is the state of $\mathscr{S}$, then $p_T^E(X)$ is the probability that a measurement of $E$ on $\mathscr{S}$ in the state $T$ will lead to a result in the set $X$. To build up a measurement theory for $E$ one usually goes along with the above ideas fixing first a *measuring apparatus* $\mathscr{A}$ (with a Hilbert space $\mathscr{H}_{\mathscr{A}}$), its *initial*

*state* $T_{\mathscr{A}} \in \mathscr{T}(\mathscr{H}_{\mathscr{A}})_1^+$, a *pointer observable* $P_{\mathscr{A}}: \mathscr{F}_{\mathscr{A}} \to \mathscr{L}(\mathscr{H}_{\mathscr{A}})$ [with its value space $(\Omega_{\mathscr{A}}, \mathscr{F}_{\mathscr{A}})$] together with a (measurable) *pointer function* $f: \Omega_{\mathscr{A}} \to \Omega$ (which correlates the pointer values with the values of the measured observable), and a *measurement coupling* $V: \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_{\mathscr{A}}) \to \mathscr{T}(\mathscr{H} \otimes \mathscr{H}_{\mathscr{A}})$ (a positive linear trace preserving mapping). The interpretation of the resulting 5-tuple $\langle \mathscr{H}_{\mathscr{A}}, P_{\mathscr{A}}, T_{\mathscr{A}}, V, f \rangle$ as a measurement of the observable $E$ starts with the idea that if $T \in \mathscr{T}(\mathscr{H})_1^+$ is the initial state of $\mathscr{S}$, then $V(T \otimes T_{\mathscr{A}})$ is the final state of the compound system $\mathscr{S} + \mathscr{A}$, with the reduced states $\mathscr{R}_{\mathscr{S}}(V(T \otimes T_{\mathscr{A}}))$ and $\mathscr{R}_{\mathscr{A}}(V(T \otimes T_{\mathscr{A}}))$ being thus the states of $\mathscr{S}$ and $\mathscr{A}$ after the measurement. Let me recall that, for example, the state $\mathscr{R}_{\mathscr{S}}(V(T \otimes T_{\mathscr{A}}))$ is defined as the partial trace of $V(T \otimes T_{\mathscr{A}})$ over the apparatus Hilbert space $\mathscr{H}_{\mathscr{A}}$; that is, as the state of $\mathscr{S}$ for which

$$\operatorname{tr}[\mathscr{R}_{\mathscr{S}}(V(T \otimes T_{\mathscr{A}}))B] = \operatorname{tr}[V(T \otimes T_{\mathscr{A}})B \otimes I] \tag{6}$$

for any $B \in \mathscr{L}(\mathscr{H})$.

The basic requirement for $\langle \mathscr{H}_{\mathscr{A}}, P_{\mathscr{A}}, T_{\mathscr{A}}, V, f \rangle$ to constitute a measurement of $E$ is derived from the interpretation of the numbers $p_T^E(X)$ as probabilities for measurement outcomes. Indeed, if $p_T^E(X) = 1$, then one would expect that the pointer observable would show this value after the measurement with certainty, that is, $\operatorname{tr}[\mathscr{R}_{\mathscr{A}}(V(T \otimes T_{\mathscr{A}}))P_{\mathscr{A}}(f^{-1}(X))] = 1$. This is the *calibration condition* of a measurement. When dealing with POV measures this condition is, however, not enough to recover the whole measurement outcome distribution $X \mapsto p_T^E(X)$ from the distribution of the pointer values after the measurement. For that end, the whole *probability reproducibility condition* is needed:

$$p_T^E(X) = p^{P_{\mathscr{A}}}_{\mathscr{R}_{\mathscr{A}}(V(T \otimes T_{\mathscr{A}}))}(f^{-1}(X)) \tag{7}$$

for all $X \in \mathscr{F}$, and for any $T \in \mathscr{T}(\mathscr{H})_1^+$.

It is obvious that a quintuple $\langle \mathscr{H}_{\mathscr{A}}, P_{\mathscr{A}}, T_{\mathscr{A}}, V, f \rangle$ satisfying condition (7) does not exhaust the physics of a measurement of an observable $E$. Further assumptions and conditions on $\langle \mathscr{H}_{\mathscr{A}}, P_{\mathscr{A}}, T_{\mathscr{A}}, V, f \rangle$ are needed; in particular, those aiming to explain the idea that a measurement leads to a definite result. Typically various correlation conditions or even the so-called objectification requirement have been studied in that context (Busch *et al.*, 1991; van Fraassen, 1991). For the present purposes the probability reproducibility condition is the most crucial (and minimal) condition for $\langle \mathscr{H}_{\mathscr{A}}, P_{\mathscr{A}}, T_{\mathscr{A}}, V, f \rangle$ to constitute a measurement of $E$. Therefore, for the sake of terminological simplicity, any quintuple $\langle \mathscr{H}_{\mathscr{A}}, P_{\mathscr{A}}, T_{\mathscr{A}}, V, f \rangle$ which fulfills this condition is called here a *measurement* of $E$, and it is denoted simply as $\mathscr{M}$. For a more extensive study of the measurement process as presented above, see, for instance, Busch *et al.* (1991), where references to many important original contributions can also be found.

Before describing the basic results of the quantum theory of measurement I shall introduce an important subclass of measurements.

Consider a measurement $\mathcal{M}$ of an observable $E$. This measurement is *unitary* if the measurement coupling $V$ is given by a unitary operator $U$, that is, $V(T \otimes T_{\mathcal{A}}) = U(T \otimes T_{\mathcal{A}})U^*$ for some unitary mapping $U$ acting on $\mathcal{K} \otimes \mathcal{K}_{\mathcal{A}}$, and it is *normal* if the pointer observable $P_{\mathcal{A}}$ is a PV measure with the same value space and scale as $E$, and if the initial state of $\mathcal{A}$ is a vector state $\phi$. A normal unitary measurement of $E$ is denoted by $\mathcal{M}_U$. Clearly, the von Neumann model $\langle \mathcal{K}_{\mathcal{A}}, Z, \phi, U_{vNL} \rangle$ is a normal unitary measurement of $A = \sum a_i P_i$. In fact, any unitary mapping

$$U: \quad \varphi \otimes \phi \mapsto \sum \langle \varphi_{ij} | \varphi \rangle \psi_{ij} \otimes \phi_i \tag{8}$$

with $(\psi_{ij}) \subset \mathcal{K}$, $\|\psi_{ij}\| = 1$, $\langle \psi_{ij} | \psi_{ij'} \rangle = \delta_{jj'}$, defines together with $\mathcal{K}_{\mathcal{A}}$, $\phi$, and $Z$ another normal unitary measurement of $A$ (Beltrametti *et al.*, 1990).

## 2.2. Basic Results

Consider, again, a measurement $\mathcal{M}$ of an observable $E$. It defines a transformation of the states of the system conditionalized with the measurement outcome. This is most conveniently expressed by the *instrument* induced by $\mathcal{M}$, which is just a state transformation valued measure $\mathcal{I}_{\mathcal{M}} : \mathcal{F} \to \mathcal{L}(\mathcal{T}(\mathcal{K}))^+$ given by

$$\mathcal{I}_{\mathcal{M}}(X)T := \mathcal{R}_{\mathcal{S}}[V(T \otimes T_{\mathcal{A}}) \cdot I \otimes P_{\mathcal{A}}(f^{-1}(X))] \tag{9}$$

for all $X \in \mathcal{F}$, $T \in \mathcal{T}(\mathcal{K}_{\mathcal{S}})_1^+$. Clearly, this instrument is compatible with the measured observable, that is

$$p_T^E(X) = \mathrm{tr}[\mathcal{I}_{\mathcal{M}}(X)T], \qquad X \in \mathcal{F}, \quad T \in \mathcal{T}(\mathcal{K}_{\mathcal{S}})_1^+ \tag{10}$$

This means that the measurement determines uniquely the measured observable. Moreover, the instrument $\mathcal{I}_{\mathcal{M}}$ gives the transformation of the state of the system under the measurement: $T \mapsto \mathcal{I}_{\mathcal{M}}(X)T$, where $\mathcal{I}_{\mathcal{M}}(X)T$ is the (nonnormalized) state of $\mathcal{S}$ after the measurement on the condition that the measurement led to a result in the set $X$. Strictly speaking, the condition here refers, in the first instance, to the value $f^{-1}(X)$ of the pointer observable; for details, see Cassinelli and Lahti (1992). The instrument $\mathcal{I}_{\mathcal{M}}$ thus contains all the information on the measurement $\mathcal{M}$ which relates directly to the measured system.

To illustrate the above ideas with a familiar example, let us observe that the instrument $\mathcal{I}_L^A$ defined by the measurement $\langle \mathcal{K}_{\mathcal{A}}, Z, \phi, U_{vNL} \rangle$ of $A = \sum a_i P_i$ has the form

$$P[\varphi] \mapsto \mathcal{I}_L^A(X)P[\varphi] := \sum_{a_i \in X} P_i P[\varphi] P_i \equiv \sum_{a_i \in X} p_\varphi^A(a_i) P[\varphi^{a_i}] \tag{11}$$

where $\varphi^{a_i} := P_i\varphi / p^A_\varphi(a_i)$. In a similar way, one may quickly confirm that the instrument defined by a unitary mapping (8) takes the form

$$P[\varphi] \mapsto \mathcal{I}^A_U(X)P[\varphi] := \sum_{a_i \in X} p^A_\varphi(a_i)P[\gamma_i] \tag{12}$$

where $\gamma_i$ is the normalized form of the vector $\sum_j \langle \varphi_{ij} | \varphi \rangle \psi_{ij}$.

It may well happen that two measurements $\mathcal{M}$ and $\tilde{\mathcal{M}}$ of $E$ define the same instrument, that is, $\mathcal{I}_\mathcal{M} = \mathcal{I}_{\tilde{\mathcal{M}}}$. One then says that the measurements are *equivalent*. Recalling that an instrument is completely positive if all its operations are completely positive (Davies, 1976), one may observe that the instrument $\mathcal{I}_U$ induced by a normal unitary measurement $\mathcal{M}_U$ of $E$ is completely positive. The following result, due to Kraus (1983) and Ozawa (1984), is the basic one:

*Lemma.* Any $E$-compatible, completely positive instrument $\mathcal{I}$ is of the form $\mathcal{I} = \mathcal{I}_U$ for some normal unitary measurement $\mathcal{M}_U$ of $E$.

When this result is taken together with the simple fact that for each observable $E$ there exist $E$-compatible completely positive instruments, we arrive at the fundamental existence result:

*Theorem.* For any observable there exist normal unitary measurements.

This result means that physical quantities, represented as POV measures, can be measured. It may also be taken as an excuse to use the blamed term observable instead of the more neutral concept of a physical quantity (Lévy-Leblond and Balibar, 1989).

### 2.3. Repeatable Measurements

The von Neumann–Lüders measurement $\langle \mathcal{H}_\mathcal{A}, Z, \phi, U_{vNL} \rangle$ of an observable $A = \sum a_i P_i$ has a number of important properties. In particular, it is repeatable, that is, its repeated application does not lead to a new result. Indeed, the probability of obtaining a result $a_i$ upon repetition, on the condition that the preceding measurement just gave this result, equals one. This idea can be formulated in the present context in various equivalent ways, the most suggestive being the following one:

$$p^A_{\varphi^{a_i}}(a_i) = 1 \tag{13}$$

where $\varphi^{a_i}$ is the (normalized vector) state of $\mathcal{S}$ after the (first) measurement, on the condition that it led to the result $a_i$. In terms of the instrument $\mathcal{I}^A_L$ this condition reads

$$\mathrm{tr}[\mathcal{I}^A_L(a_i)\mathcal{I}^A_L(a_i)P[\varphi]] = \mathrm{tr}[\mathcal{I}^A_L(a_i)P[\varphi]] \tag{14}$$

and it is valid for any initial state of the system and for all possible values of $A$.

Since the pioneering work of von Neumann (1932) it has been an important problem to find out the conditions under which a measurement is repeatable. This problem has now been essentially solved.

Consider a measurement $\mathcal{M}$ of an observable $E$. In accordance with the above ideas, $\mathcal{M}$ is *repeatable* if its instrument $\mathcal{I}_{\mathcal{M}}$ is repeatable, that is

$$\mathrm{tr}[\mathcal{I}_{\mathcal{M}}(X)\mathcal{I}_{\mathcal{M}}(X)T] = \mathrm{tr}[\mathcal{I}_{\mathcal{M}}(X)T] \tag{15}$$

for all $T \in \mathcal{T}(\mathcal{H})_1^+$ and for all $X \in \mathcal{F}$. [For a more detailed analysis of this concept, see, e.g., Busch *et al.* (1991) and Lahti *et al.* (1991).]

The following result gives a necessary condition for $\mathcal{M}$ to be repeatable.

*Theorem.* If a measurement is repeatable, then the measured observable is discrete.

*Remark.* This result presumes that the value space $(\Omega, \mathcal{F})$ of the instrument $\mathcal{I}_{\mathcal{M}}$ is a standard Borel space, an assumption which seems to exclude no physically relevant cases. In particular, the real Borel spaces $(\Re^n, \mathcal{B}(\Re^n))$ are such. The proof of the theorem is due to Łuczak (1986). Ozawa (1984) also proved the same result under the additional assumption that the measurement is equivalent to a unitary measurement.

Clearly, a measurement of a discrete observable, like $A = \sum a_i P_i$, need not be repeatable. On the other hand, any discrete observable represented as a PV measure admits a repeatable measurement, for instance, a von Neumann–Lüders measurement. Sufficient conditions for a discrete observable $E: \omega_i \to E_i$ to admit a repeatable measurement can be formulated (Cassinelli and Lahti, 1992; Ozawa, 1984; Łuczak, 1986). What is important, however, is the above-quoted result, which states that an observable must be discrete in order to admit a repeatable measurement.

In concluding this Section, I shall describe conditions on measurements which imply that they are equivalent to the prototype measurement of von Neumann.

Another important property of the von Neumann–Lüders measurement is its ideality, that is, it changes the state of the system only to the extent necessary for the measurement outcome. There are again various alternative formulations of this notion. However, in the context of discrete observables $\omega_i \mapsto E_i$ the most obvious one is perhaps the following:

$$\text{if} \quad p_T^E(\omega_i) = 1, \quad \text{then} \quad \mathcal{I}_{\mathcal{M}}(\omega_i)T = T \tag{16}$$

For discrete observables represented as PV measures the ideality of a measurement already implies that the measurement is equivalent to a von Neumann–Lüders measurement; see, for instance, Lahti *et al.* (1991). Thus, a repeatable ideal measurement of an observable represented by a PV measure is equivalent to a von Neumann–Lüders measurement. For observables which are given as POV measures an additional technical requirement is needed. This is the nondegeneracy or faithfulness assumption. A measurement $\mathcal{M}$ is *faithful* if its instrument $\mathcal{I}_{\mathcal{M}}$ is faithful, that is, if for any $B \in \mathcal{L}(\mathcal{H})$, the conditions $\mathrm{tr}[\mathcal{I}_{\mathcal{M}}(X)TB] = 0$, $X \in \mathcal{F}$, and $T \in \mathcal{T}(\mathcal{H})_1^+$ imply that $B = O$ (Davies, 1976). Indeed, if an observable admits a repeatable faithful measurement, then the observable is a discrete PV measure (Davies, 1976; Łuczak, 1986). Hence we may conclude with the following statement:

*Theorem.* An ideal repeatable faithful measurement is equivalent to a von Neumann–Lüders measurement.

## 3. SEQUENTIAL MEASUREMENTS

The theory of measurement presented above is important in at least two aspects. It lends itself directly to a systematic study of various types of measurements, and it allows one to investigate in a satisfactory way the interpretational issues of quantum mechanics. However, it is its derivative, the theory of sequential measurements, which has direct experimental consequences. I shall now recall those parts of this theory which are needed for its application to a problem of a geometric phase.

### 3.1. General Theory

Consider any two measurements $\mathcal{M}_1$ and $\mathcal{M}_2$ of observables $E_1$ and $E_2$, and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be their instruments. These measurements may be combined to yield a sequential measurement of the two observables in question performing them on $\mathcal{S}$ one after the other in either order. Without entering into the technical details [for which see Davies and Lewis (1970) and Busch *et al.* (1990)], let $\mathcal{M}_{12}$ stand for the sequential measurement obtained by performing first the $E_1$-measurement $\mathcal{M}_1$ and then, in immediate succession, the $E_2$-measurement $\mathcal{M}_2$. Let $\mathcal{I}_{12}$ denote the resulting instrument. One may show that this instrument is, in fact, the composition of the instruments $\mathcal{I}_1$ and $\mathcal{I}_2$, that is, $\mathcal{I}_{12}(X \times Y)T = \mathcal{I}_2(Y)(\mathcal{I}_1(X)T)$ for all value sets $X \in \mathcal{F}_1$ and $Y \in \mathcal{F}_2$ and for all states $T \in \mathcal{T}(\mathcal{H})_1^+$.

The probability that the sequential $E_1E_2$-measurement on $\mathcal{S}$ in the state $T$ leads to a result in $(X, Y)$ is

$$\mathrm{tr}[\mathcal{I}_2(Y)(\mathcal{I}_1(X)T)] = p_T^{E_1}(X) \cdot p_{T_X}^{E_2}(Y) \tag{17}$$

where $T_X \equiv \mathscr{I}_1(X)T/p_T^{E_1}(X)$ is the state of $\mathscr{S}$ after the first measurement on the condition that it led to a result in $X$. These probabilities are called *sequential probabilities*. They depend explicitly on the first measurement. Thus, if one is able to perform, at all, sequential measurements, such measurements would immediately serve also as tests between the applied measurement models. Dicke (1989) proposes a realistic sequential photon polarization experiment. It still remains to be seen whether such measurements can in fact be carried out. As to the sequential probabilities, it is to be noted that they do not depend on whether the conditioning, that is, the reading of results, is done stepwise or only terminally.

The sequential outcome $(X, Y)$ is accompanied by a state transformation

$$T \mapsto \mathscr{I}_{12}(X \times Y)T = \mathscr{I}_2(Y)(\mathscr{I}_1(X)T) \tag{18}$$

The final state of $\mathscr{S}$—with the condition $(X, Y)$—depends on the two measurements, but again it does not depend on the conditioning whether it is stepwise or not. However, with intermediate conditioning one would get

$$T \mapsto \mathscr{I}_1(X)T \mapsto \mathscr{I}_2(Y)(\mathscr{I}_1(X)T)$$

Consider now a (finite) sequence of measurements of the observables

$$A^s = \sum a_i^s P_i^s, \qquad s = 1, \ldots, n \tag{19}$$

on the system prepared initially in a pure state $P[\varphi]$, and assume that the involved measurements are of the type (8). The relevant state transformations are then of the form (12). The probability that such a sequential measurement of the observables $A^1, A^2, \ldots, A^n$ leads to an outcome sequence $(a_{k_1}^1, a_{k_2}^2, \ldots, a_{k_n}^n)$ is then simply

$$\begin{aligned} &p_\varphi^{A^1}(a_{k_1}^1) \cdot p_{\gamma_{k_1}^1}^{A^2}(a_{k_2}^2) \cdots p_{\gamma_{k_{n-1}}^{n-1}}^{A^n}(a_{k_n}^n) \\ &\quad = \langle \varphi | P_{k_1}^1 \varphi \rangle \langle \gamma_{k_1}^1 | P_{k_2}^2 \gamma_{k_1}^1 \rangle \cdots \langle \gamma_{k_{n-1}}^{n-1} | P_{k_n}^n \gamma_{k_{n-1}}^{n-1} \rangle \end{aligned} \tag{20}$$

Such an outcome is accompanied by the state change

$$P[\varphi] \to P[\gamma_{k_n}^n] \tag{21}$$

which can formally be decomposed as

$$P[\varphi] \to P[\gamma_{k_1}^1] \to \cdots \to P[\gamma_{k_n}^n] \tag{22}$$

This sequence of state changes is, however, not given by the above sequential $A^1 \cdots A^n$-measurement with the result $(a_{k_1}^1, \ldots, a_{k_n}^n)$, but it is subject to a further assumption, namely *sequential conditioning*. This is the assumption that after each involved $A^s$-measurement the conditioning with respect to the pointer value $z_{k_s}$ is obtained. In other words, the sequential state change (22) is obtained in the course of the sequential $A^1 \cdots A^n$-mea-

surement if after each $A^s$-measurement the measurement outcome $z_{k_s}$ is registered. It is to be emphasized that in both cases (intermediate conditioning or not) the probability of obtaining the outcome sequence $(a^1_{k_1}, \ldots, a^n_{k_n})$ as well as the final (conditional) state $P[\gamma^n_{k_n}]$ of the system are the same; they do not depend on the type of conditioning whether obtained stepwisely or only terminally. To obtain also the intermediate states $P[\gamma^s_{k_s}]$ requires, however, intermediate conditionings.

For the application in mind, let us close the above measurement sequence with a measurement of the simple observable

$$A^0 = a^0_+ P[\varphi] + a^0_- (I - P[\varphi]) \tag{23}$$

choosing a $U^0$-generating set $(\psi^0_{ij})_{i,j \geq 1}$ in (8) such that $\psi^0_1 = \varphi$. The cyclic state change

$$P[\varphi] \to P[\varphi] \tag{24}$$

is then associated with the outcome sequence $(a^1_{k_1}, \ldots, a^n_{k_n}, a^0_+)$, the probability of which is

$$\begin{aligned} &p^{A^1}_{\varphi}(a^1_{k_1}) \cdot p^{A^2}_{\gamma^1_{k_1}}(a^2_{k_2}) \cdots p^{A^0}_{\gamma^n_{k_n}}(a^0_+) \\ &\quad = \langle \varphi | P^1_{k_1} \varphi \rangle \langle \gamma^1_{k_1} | P^2_{k_2} \gamma^1_{k_1} \rangle \cdots \langle \gamma^n_{k_n} | P[\varphi] \gamma^n_{k_n} \rangle \end{aligned} \tag{25}$$

With the assumption of sequential conditioning the state change (24) decomposes into a cycle

$$P[\varphi] \to P[\gamma^1_{k_1}] \to \cdots \to P[\gamma^n_{k_n}] \to P[\varphi] \tag{26}$$

### 3.2. Geometric Phase

It has been argued in the literature (Samuel and Bhandari, 1988; Anandan and Aharonov, 1988) that the measurement process like (26) gives rise to an experimentally measurable geometric phase. The development of a geometric phase in the course of such a process remains, however, undefined, since (26) gives only the $n + 1$ points in the projective space of the pure states, and not any curve connecting these points. To define a geometric phase for the process (26), one may proceed as follows (Cassinelli *et al.*, 1992). Let

$$\bar{C}: \quad \varphi \to e^{i\alpha_1} \gamma^1_{k_1} \to \cdots \to e^{i\alpha_n} \gamma^n_{k_n} \to e^{i\alpha} \varphi \to \varphi \tag{27}$$

be any closed curve of unit vectors in $\mathscr{H}$ such that its canonical projection $\pi(\bar{C})$ on the projective space of pure states consists of geodesic lines connecting the succeeding states $P[\varphi]$, $P[\gamma^1_{k_1}], \ldots, P[\gamma^n_{k_n}]$, and $P[\varphi]$. Define

$$\Gamma := -\mathrm{i} \oint_{\bar{C}} \langle \varphi | d\varphi \rangle \tag{28}$$

It can be shown that $\Gamma$ does not depend on the phases of the involved vectors, but only on the geodesic lines connecting the states in (26). The assumption that the *geometric phase* accumulated in the measurement process (26) is given by $\Gamma$ is referred to as the *geodesic hypothesis* (Cassinelli *et al.*, 1992). It is to be emphasized that a sequential measurement process like (24) does not define in itself a geometric phase. To obtain that, some supplementary assumptions, here the sequential conditioning and the geodesic hypothesis, are always needed.

The phase $\Gamma$ can be calculated explicitly, and one gets

$$\Gamma = \arg(\langle \varphi | P[\gamma^1_{k_1}] \cdots P[\gamma^n_{k_n}] | \varphi \rangle \tag{29}$$

where $\arg(c)$ is the argument of the complex number $c$ (Cassinelli *et al.*, 1992). From this expression it is clear that the geometric phase depends only on the *states* $P[\gamma^i_{k_i}]$ and not on the *vectors* $\gamma^i_{k_i}$.

It is evident that $\Gamma$ depends on the two assumptions, the sequential conditioning and the geometric hypothesis, but also on the involved measurements. If the applied $A^s$-measurements were von Neumann–Lüders measurements, one would get

$$\Gamma_{vNL} = \arg(\langle \varphi | P^{(n)}_{k_n} \cdots P^{(1)}_{k_1} \varphi \rangle) \tag{30}$$

which, in general, differs from (29).

An experimental determination of the geodesic phase in a cyclic sequential measurement process would thus provide a test of (1) the validity of the geodesic hypothesis, (2) the validity of the assumption on the sequential conditioning, and (3) the validity of the applied measurement model. The possibility of an experimental determination of the geometric phase in sequential measurements is anticipated in Samuel and Bhandari (1988), whereas Dicke (1989) proposes a realistic model for sequential photon polarization measurements.

## REFERENCES

Anandan, J., and Aharonov, Y. (1988). Geometric quantum phase and angles, *Physical Review D*, **38**, 1863–1870.

Beltrametti, E., and Cassinelli, G. (1981). *The Logic of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Beltrametti, E., Cassinelli, G., and Lahti, P. (1990). Unitary measurements of discrete quantities in quantum mechanics, *Journal of Mathematical Physics*, **31**, 91–98.

Busch, P., Cassinelli, G., and Lahti, P. (1990). On the quantum theory of sequential measurement, *Foundations of Physics*, **20**, 757–775.

Busch, P., Lahti, P., and Mittelstaedt, P. (1991). *The Quantum Theory of Measurement*, Springer, Berlin.

Cassinelli, C., and Lahti, P. (1992). Conditional probability in quantum theory of measurement, *Nuovo Cimento B*, **108**, 45–56.

Cassinelli, G., de Vito, E., Lahti, P., and Levrero, A. (1992). Geometric phase and sequential measurements in quantum mechanics, *Physical Review A*, submitted.

Davies, E. B. (1976). *Quantum Theory of Open Systems*, Academic Press.

Davies, E., and Lewis, J. (1970). An operational approach to quantum probability, *Communications in Mathematical Physics*, **17**, 239–259.

Dehmelt, H. (1990). Less is more: Experiments with an individual atomic particle at rest in free space, *American Journal of Physics*, **58**, 17–27.

Dicke, R. (1989). Quantum measurements, sequential and latent, *Foundations of Physics*, **19**, 385–395.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.

Kraus, K. (1983). *States, Effects, and Operations*, Springer, Berlin.

Lahti, P., Busch, P., and Mittelstaedt, P. (1991). Some important classes of quantum measurements and their information gain, *Journal of Mathematical Physics*, **32**, 2770–2775.

Lévy-Leblond, J.-M., and Balibar, F. (1989). *Quantics*, North-Holland, Amsterdam.

Łuczak, A. (1986). Instruments on von Neumann algebras, *Acta Universitatis Lodziensis*.

Mittelstaedt, P. (1993). Quantum measurement I, *International Journal of Theoretical Physics*, this issue.

Ozawa, M. (1984). Quantum measuring process of continuous observables, *Journal of Mathematical Physics*, **25**, 79–87.

Samuel, J., and Bhandari, R. (1988). General setting for Berry's phase, *Physical Review Letters*, **60**, 2339–2342.

Van Fraassen, B. C. (1991). *Quantum Mechanics: An Empiricist View*, Clarendon, Oxford.

Von Neumann, J. (1932). *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin.

Wheeler, J., and Zurek, W. (1983). *Quantum Theory and Measurement*, Princeton University Press, Princeton, New Jersey.